“Palindromic” and “Quasicrystalline” Characteristics of the Fibonacci Sequence and Words.

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Abstract:

This article describes substitution rules (or morphisms), which from a binary alphabet $\boldsymbol{A}$ = {L,S} generate a series of palindromes the grand total of which is the infinite Fibonacci word. It also shows a geometric construction, in keeping with the same development rules, producing a quasicrystal possessing two dimensions of pentagonal and decagonal symmetries.

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Introduction :

In combinatorial mathematics, the infinite Fibonacci word is constructed according to substitution rules (or morphisms) [1, 2, 3]. The arithmetical characteristics of the Fibonacci words will generate the sequence of Fibonacci numbers which are a direct consequence of it. These two developments (the words, the numbers) obey the same rules ; that is, each term equals the sum of the previous two terms, hence : $U_n = U_{n-1} + U_{n-2}$ (as far as the words are concerned, concatenation is equivalent to addition). When we give to the two letters making up the alphabet of the Fibonacci words the respective values : $S = \Phi = (1+\sqrt{5})/2$ (i.e. the golden number ; where Φ is pronounced phi [4]) and $L = \Phi^2 = \Phi + 1$, the ratio of the length of the successive words $|m_n|$ is precisely in the proportion Φ : $|m_n| / |m_{n-1}| = \Phi$. The sequence of Fibonacci numbers gives only an approximation of it [5] ; the ratio between two successive Fibonacci numbers, i.e. F_{n+1}/F_n, tends to Φ when (n) grows until reaching it at last at its infinite limit. What is the reason of this apparent dichotomy ?...

This article offers a new interpretation of these sequences and is based on a geometric model [6] aiming at restoring harmony between three systems obeying the same development rules, i.e. : $U_n = U_{n-1} + U_{n-2}$; a fractal system of circles covered by a "pentagrid" [7] ; the Fibonacci words ; the sequence of Fibonacci numbers. It manages to do so thanks to substitution rules which, from original quanta, produce only palindromes [8]. The infinite sum of these palindromes is the infinite Fibonacci word. The fractal system of circles shows us a point of origin (O.), at infinity but absolutely locatable, around which everything seems to organize itself harmoniously. After the three systems (circles, Fibonacci words and sequence) have accomplished ten 36° rotations centered on origin (O.), we get a perfect two-dimensional quasicrystal [9].

A reminder of some rules :

If we set an alphabet $\mathcal{A}$ with two letters L and S, the sequence (m_n) of Fibonacci words is defined by : $m_1 = S$ and $m_2 = L$ and for n > 2 by $m_n = m_{n-1}\, m_{n-2}$ (the product is the concatenation of the previous two terms).

The infinite Fibonacci word, written M_∞, is the limit of this sequence, that is to say, the unique infinite word for which all words m_n are prefixes. The Fibonacci words are so called by analogy with the Fibonacci numbers, since the construction process is analog, addition being replaced with concatenation.

The first Fibonacci words and numbers are :

Fibonacci words :	Fibonacci numbers :
m_1 = S	1
m_2 = L	1
m_3 = LS	2
m_4 = LSL	3
m_5 = LSLLS	5
m_6 = LSLLSLSL	8
m_7 = LSLLSLSLLSLLS	13
m_8 = LSLLSLSLLSLLSLSLLSLSL	21
Etc.	Etc.

The right column, which shows the number of letters in each of the words m_n, forms the Fibonacci sequence. The Fibonacci words can also be generated by a "morphism," in other words, by substitution rules applied iteratively.

Given a word "m" in letters S and L, we transform it by replacing each S by L and each L by LS, then we iterate the process. Let us now discuss the word m_1 = S. By applying the substitutions described S → L and L → LS, we get m_2 = L, m_3 = LS, m_4 = LSL, etc.

These are the substitution rules commonly accepted for the construction of the infinite Fibonacci word.
We notice that only m_4 is a palindrome.

New “palindromic” substitution rules :

(Throughout this article, we will attribute the following values to S and L :
$S = \Phi = (1+\sqrt{5})/2 \approx 1.6180339887...$ (i.e. the golden number or ratio).
$L = \Phi^2 = \Phi + 1 \approx 2.6180339887...$ (i.e. the golden number or ratio squared).
We will call p_n the palindromic words produced by these new substitution rules).
Now, let us change the above substitution rules by the following rules :
S → LSL, L → LSLSL. By starting from p_1 = S, and applying iteratively these new rules (that is, an amplification factor Φ^3 to each iteration when $S = \Phi$ and $L = \Phi^2$), if we carry the process further, we get :
p_1 = S = 1 letter
p_2 = LSL = 3 letters
p_3 = LSLSLLSLLSLSL = 13 letters
p_4 = LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSL = 55 letters
p_5 =
LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSL
LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSL
LSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSL
LSLLSLSLLSLSLLSLLSLSLLSLLSLSL = 233 letters
p_6 = 987 letters
p_7 = 4,181 letters
p_8 = 17,711 letters
Etc.
We notice that :
- from p_1=S, which is the original quantum, these substitution rules will systematically produce palindromes at the following iteration as a consequence of the principle of symmetry and therefore for each of the following words p_n ; S always taking up the center of these palindromes ; L always being at their extremities.
- each of the palindromic words p_n contains a number of letters corresponding to a Fibonacci number out of three starting from the first 1 of the Fibonacci sequence.
- p_2 being the beginning of the infinite Fibonacci word and a palindrome itself, the infinite Fibonacci word in full is a sum of palindromes. We will therefore be able to write the infinite Fibonacci word as follows :

(With M_∞ for the infinite Fibonacci word ; $n \in N^*$; σ the morphism so that S → LSL and L → LSLSL ; and p_n the palindromes generated by this morphism).

$$M_\infty = \sum_{n=1}^{\infty} \sigma p_n$$

- in this case, the infinite Fibonacci word M_∞ can be reached thanks to the concatenation, in ascending order, of all the previous palindromic words p_n except p_1 ; that is :
$M_\infty = p_2\, p_3\, p_4\, p_5\, p_6\, p_7\, p_8\, p_9\, p_{10}\, p_{11}\, p_{12}\, p_{13}\, p_{14}\, p_{15}\, p_{16}\, p_{17} \ldots$ etc. p_∞ .
This property will result in a greater physical distance between p_n $(n > 1)$ and p_1 when (n) grows.
- here, “p_∞” is not the infinite Fibonacci word, but the infinite palindrome generated by $p_{\infty-1}$.

- if we call $p_{pref.n}$ the prefix factor which separates p_n from p_1, we can confirm that p_{n+1} will start with a factor equivalent but chiral* of $p_{pref.n}$ that we will call $p_{suf.n}$ suffix of p_n. Therefore, the factor formed by $p_{pref.n}\, p_n\, p_{suf.n}$ is a palindrome. (*Said of an object which is not superimposable on its image in a mirror plane.)
- the distance between "p_∞" and p_1 is the factor $p_{pref.\infty} = p_2\, p_3\, p_4\, p_5\, p_6\, p_7\, p_8\, p_9\, p_{10} \ldots$ etc. $p_{\infty-1}$.
- "$p_{\infty+1}$" will start with a factor $p_{suf.\infty}$ chiral of $p_{pref.\infty}$.
- therefore, there is a palindrome formed by "$p_{pref.\infty}\, p_\infty\, p_{suf.\infty}$"; this palindrome covers the infinity of the one-dimensional space and we will call it P_∞.
- the infinite Fibonacci word $M_\infty = p_{pref.\infty} + p_\infty$ and $P_\infty = M_\infty + p_{suf.\infty}$. This configuration allows us to know the infinite extremity ($p_{suf.\infty}$) as perfectly as its beginning ($p_{pref.\infty}$), the expansion of the successive words p_n occurring symmetrically from the center of the palindromes.

Table A : development of the Fibonacci sequence and palindromic words :

Column 1 : names of the palindromic words.
Column 2 : number of letters in the words (analogous to the Fibonacci sequence).
Column 3 : palindromic Fibonacci words.
Column 4 : metric lengths of the words $|p_n|$ when S = Φ and L = Φ^2.
Column 5 : distance from the origin (O.) of the beginning of each word $|p_n|$ (see Figure.1).

1	2	3	4	5
p_1	1	S	Φ	1/2
Φp_1	1	L	Φ^2	$\Phi/2$
$\Phi^2 p_1$	2	½SLS½	Φ^3	$\Phi^2/2$
p_2	3	LSL	Φ^4	$\Phi^3/2$
Φp_2	5	½LSLLSL½	Φ^5	$\Phi^4/2$
$\Phi^2 p_2$	8	½SLLSLSLLS½	Φ^6	$\Phi^5/2$
p_3	13	LSLSLLSLLSLSL	Φ^7	$\Phi^6/2$
Φp_3	21	½LSLLSLLSLSLLSLSLLSLLSL½	Φ^8	$\Phi^7/2$
$\Phi^2 p_3$	34	½SLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLS½	Φ^9	$\Phi^8/2$
p_4	55	LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLL SLLSLSL	Φ^{10}	$\Phi^9/2$
Φp_4	89	½LSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLS LSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSL½	Φ^{11}	Φ^{10} /2
$\Phi^2 p_4$	144	½SLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSL SLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLS LLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSL LS½	Φ^{12}	$\Phi^{11}/2$
p_5	233	LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLL SLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLS LLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSL LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLL SLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSL	Φ^{13}	$\Phi^{12}/2$
Etc.	Etc.	Etc.	Etc.	Etc.
p_∞	∞	LSLSLLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLL SLLSLSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLSLSLLSLS... ∞	Φ^∞	$\Phi^{\infty-1}/2$

Interpretation of table A and consequences for the Fibonacci sequence :

Of course, in theory, the letters making up the alphabet of the Fibonacci words are indivisible and the sequence of the words p_n is enough to constitute the entire infinite Fibonacci word with whole letters. The sequences Φp_n and $\Phi^2 p_n$ shown in table A are superposed on p_n with a gap of order Φ and Φ^2 respectively ; they display half letters at the extremities of their palindromes. It might not be necessary to describe them, but they are still completely in keeping with 2/3 of the Fibonacci numbers and they are part of the construction logic of an underlying fractal model (figure 1).

These two series of words, made up of the Φp_n and $\Phi^2 p_n$, correspond to the Fibonacci numbers relinquished by the sequence of the p_n , they obey the same substitution rules as for the p_n .

The sequence Φp_n will produce palindromes having LL as their central letters and ½L, L½, at their extremities. At the next iteration, ½L will give ½LSL which is the right half of the substitution LSLSL and L½ will give LSL½ which is the left half of LSLSL. The sequence $\Phi^2 p_n$ corresponds to the sequence of even Fibonacci numbers (1 out of 3) and will produce palindromes having L as their central letter, ½S and S½ at their extremities. At the next iteration, ½S will give ½SL which is the right half of the substitution LSL and S½ will give LS½ which if the left half of LSL.

When we observe table A, we notice that the content of each line is multiplied by a factor Φ on the next line, except in column 2 since it contains natural numbers counting the number of S and L in each word p_n without making any qualitative distinction between these two letters. We then understand that the second number 1 of the Fibonacci sequence cannot have the same value as the first and that it designates an entity Φ times greater. All the mathematical literature dedicated to the Fibonacci sequence explains that the ratio of each term of the sequence to the previous one comes closer to Φ without ever reaching this value (except at its infinite limit) ; this is true if we consider the sequence from a strictly arithmetical point of view, but this amounts, from a physical perspective and according to the French idiom, to “get cabbage and carrots mixed up” ; or from Fibonacci’s point of view, “young and adult rabbits.” Even more philosophically and metaphysically, the first young rabbits cannot come from scratch or arrive from nowhere; consequently, starting the Fibonacci sequence with 0, 1, 1… is nonsensical.

The Fibonacci sequence as we usually write it : 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … is not a geometric sequence of common ratio q = Φ quite simply because it starts from zero, and because, for the calculations inherent in this sequence, we attribute the same value to the second 1 as to the first. The ratios between two successive terms in this sequence are therefore inconstant.

For the sake of rigor, we should combine two sets of natural numbers ; for example :

Given ℕ the set of natural numbers counting units 1.

Given $\mathbb{N}^{\Phi}$ the set of natural numbers counting units Φ.

The black color would represent the elements of ℕ, and the red color the elements of $\mathbb{N}^{\Phi}$.

The Fibonacci sequence would take this form :

1, 1, 1+1, 1+2, 2+3, 3+5, 5+8, 8+13, 13+21, 21+34, 34+55, 55+89, 89+144, etc. + etc. ∞

Another interpretation even closer to physical reality would take this form :

-∞ …1/(13L+8S), 1/(8L+5S), 1/(5L+3S), 1/(3L+2S), 1/(2L+S), 1/(L+S), 1/L, 1/S, S/S, S, L, L+S, 2L+S, 3L+2S, 5L+3S, 8L+5S, 13L+8S, 21L+13S, 34L+21S, 55L+34S, 89L+55S… +∞.

We can infer from the sequence represented this way that there is a common ratio q so that :

q.S/S = S thus q = S, and q.S = L thus L = S^2, and so q.L = L+S.

We therefore have a geometric sequence with common ratio q = S = Φ = (1+√5)/2.

This last sequence is fractal, it has no beginning and no end, its underlying geometric structure is rigorously identical for each of its iterations ; the Fibonacci words appear and are superposed harmoniously on the structure, as soon as we set a landmark (or frame of reference) by considering that one of its iterations represents the unit or quantum of a system.

When we admit the fractal nature of a sequence, we also have to accept that its unit or quantic representative is itself divisible to infinity. Choosing or determining a quantum is therefore inevitably arbitrary (anything having acquired a consistency is potentially divisible) ; rather than demonstrating that there are physical limits, this action reveals eventually the inability of our technical, speculative and intellectual means to probe, or simply imagine, the "even smaller" (or greater).
Since the dawn of time, when human beings improve their means of observation, they discover unsuspected universes beyond what they thought were the limits; it is therefore almost certain that the known constituents of the observable Universe, which happens to have considerably stretched out over the last decades for that matter, only represent, after all, a few terracings in an infinite fractal system.

Graphic and geometric representation :

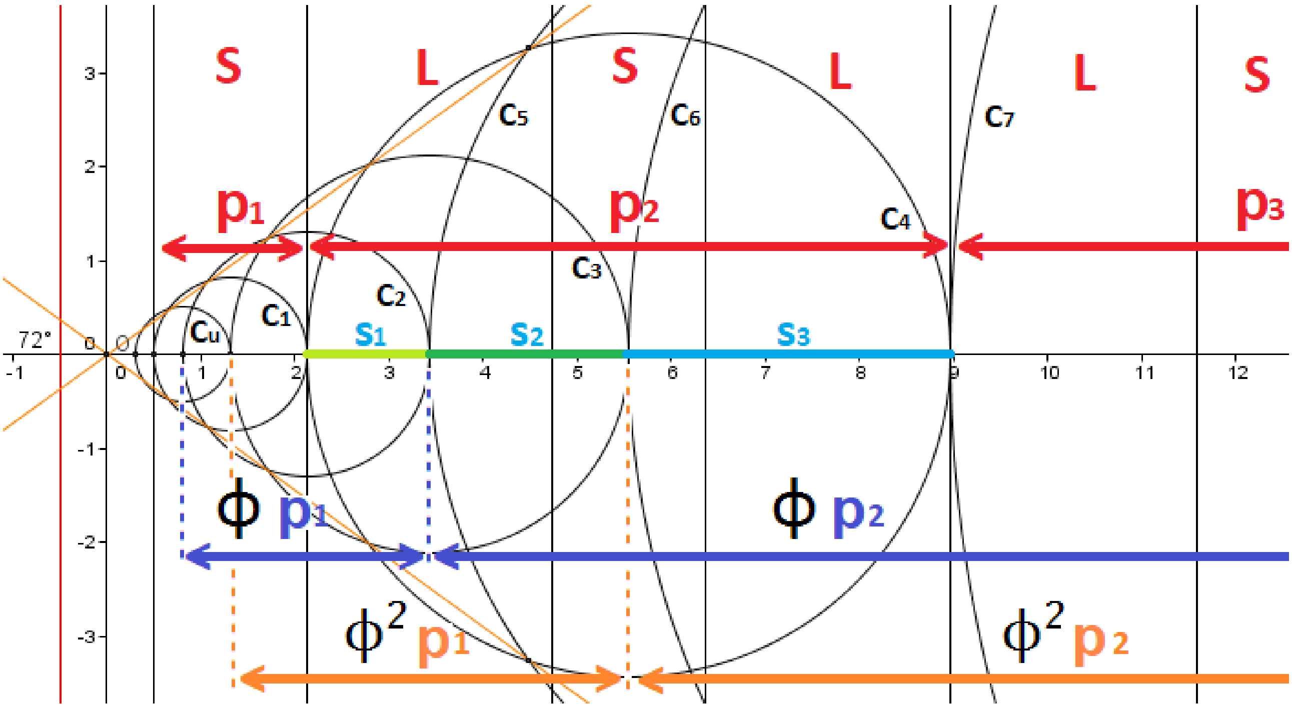

Figure 1.

Figure 1 shows a geometric progression of circles (C_1, C_2, C_3, C_4, C_5, C_6, C_7, …, C_n) ; this progression is a geometric sequence so that : $C_n = q.C_{n-1}$ with $q = \Phi$. The first circle C_u is the unit circle with diameter 1. This sequence tends to -∞ and decreases towards the origin (O.), it tends to +∞ while growing in the opposite direction. This graphic representation is therefore absolutely in keeping with the previous postulate. The diameter of each circle Cn is divided into three segments S_n , S_{n+1} , S_{n+2} , by the tangent circles $C_{n-2}\,C_{n+1}$ and $C_{n-1}\,C_{n+2}$. The progression of the segments S_n on the x-axis also obeys the geometric sequence with common ratio $q = \Phi$.
The circles' diameters (C_1, C_2, C_3, …, C_n) correspond to column 4 of table A. After starting with $p_1 = \Phi = C_1$; $\Phi p_1 = \Phi^2 = C_2$; $\Phi^2 p_1 = \Phi^3 = C_3$; then each circle C_n will contain in its diameter a "palindromic" portion of the infinite Fibonacci word.
The circles are tangent to one another in a ratio of Φ^3. Therefore, we have three series of circles overlapping and corresponding to three series of words p_n , Φp_n , $\Phi^2 p_n$, which obey the same substitution rules : S gives LSL and L gives LSLSL (or an amplification factor Φ_3 when $S = \Phi$ and $L = \Phi^2$). The distance between each circle C_n and the origin is proportional to its diameter, we can calculate it with the formula :
$d_{(0.)} = C_{n-1}/2 = (C_n/\Phi)/2 = C_n/(1+\sqrt{5})$.
The proportion between a circle's diameter and its distance from the origin is therefore :
$1+\sqrt{5} \approx 3.236067977...$

Figure 1 represents, by spacings of parallel straight lines, the two components S = Φ and L = Φ^2 of the infinite Fibonacci word and the way it appears on the plane equipped with an orthonormal reference point. We can then easily convince ourselves – looking at the perfect harmony between the systems, of the circles, the sequence, the sum of the palindromic words which compose the infinite Fibonacci word – that the first terms (C_1 for the circles; 1 for the Fibonacci sequence; S for the Fibonacci words) do not suddenly come from zero but from the infinite previous terms. By giving, very arbitrarily, a unit value to one of these terms, we initiate the development of the Fibonacci sequence and infinite word. It is also important to note that the original quantum (m_1 = Φ. indivisible?...) appears at a distance ½ from the origin (O.) ; which is somewhat reminiscent of the critical line of the Riemann zeta-function.
Another important thing to mention is that the intersection points of the successive circles materialize two axes forming a 72° or 2π/5 angle. These systems will therefore be able to cover harmoniously the whole plane in a fivefold and tenfold central symmetry.

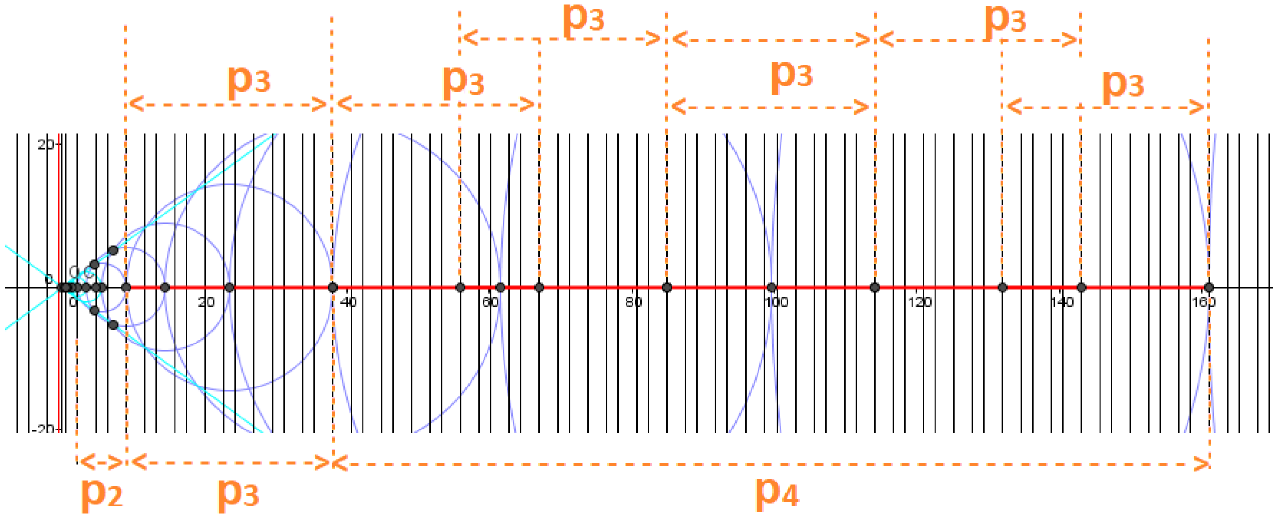

Figure 2.

Figure 2 shows how we can find p_3 in p_4 symmetrically with respect to the center of the palindrome. The same will be true for all the p_{n-1} in p_n .

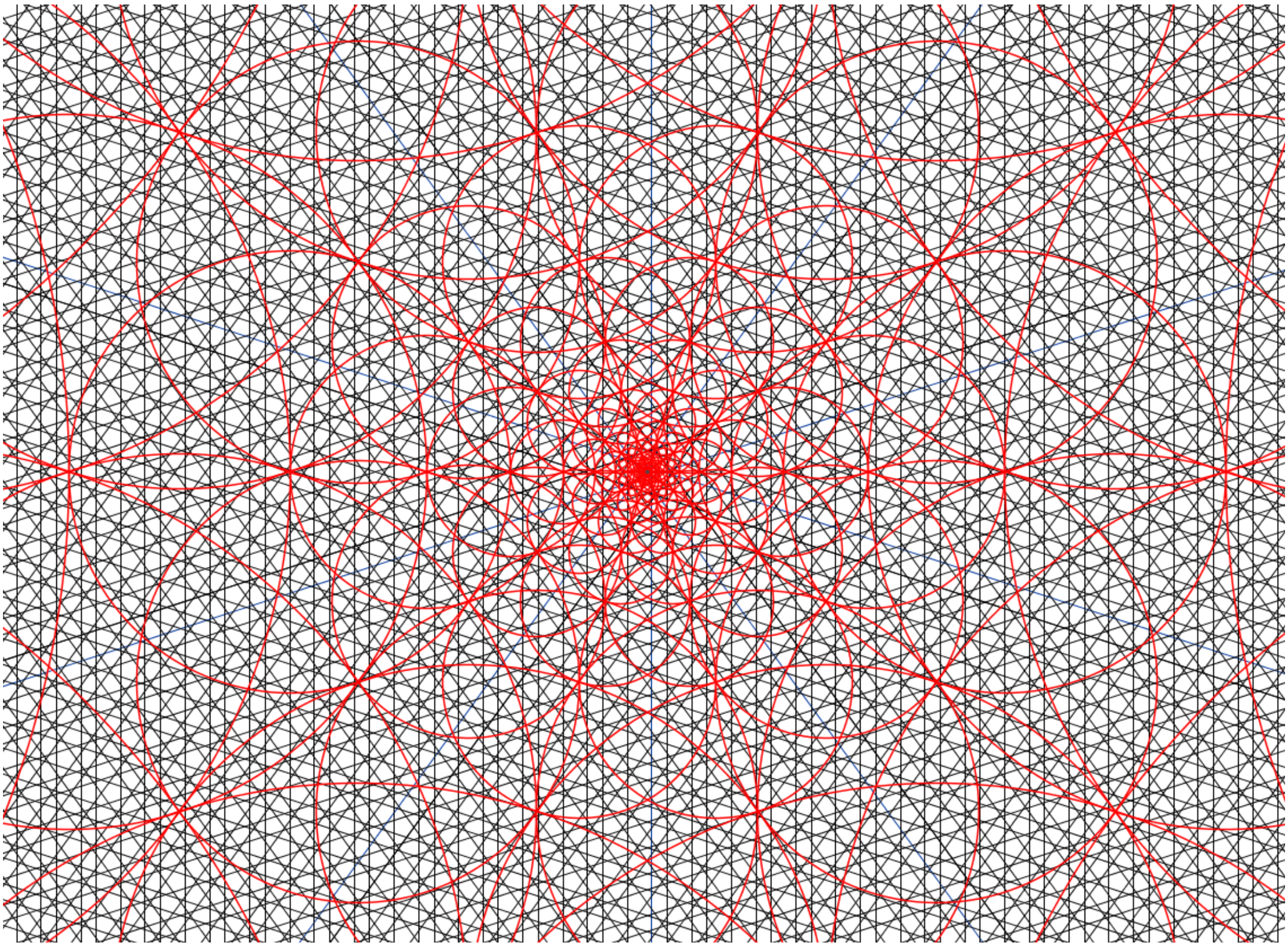

Figure 3.

Figure 3 presents the result after ten 36° rotations around the origin (O.) of the system of circles and spacings of straight lines obeying the substitution rules of the Fibonacci words. This "pentagrid" is a perfect two-dimensional quasicrystal ; it can cover the infinity of the plane, it possesses only two different types of intersections and includes only four different segment lengths without intersection. In my research, I call this "pentagrid" the "Alpha Network". The centering on the origin (O.) is necessary to obtain such a consistency. When $S = \Phi$ and $L = \Phi^2$, the four lengths are as follows (with lambda $\lambda = \sqrt{(\Phi^2 + 1)}$) :

– $i = (1/\lambda)/\Phi \approx 0.324919696... =$ tang.18°

– $h = 1/\lambda \approx 0.5257311121... = i.\Phi$

– $f = \Phi/\lambda \approx 0.8506508079... = h.\Phi$

– $d = \Phi^2/\lambda \approx 1.37638192... = f.\Phi =$ tang.54°

We notice that these lengths are also in a ratio of Φ and that we find an amplification factor Φ^3 between (i) and (d) or : tang.54°= Φ^3.(tang.18°).

This grid can also be paved or reconstituted by juxtaposing three different bricks, including a regular pentagon possessing three different states depending on whether it is crossed by one line segment or two, or none. These bricks (P0, P1, P2, the cup (Coupe) and the chalice (Calice)) make up a set which I call the "bricks of creation." They are described by Figure 4.

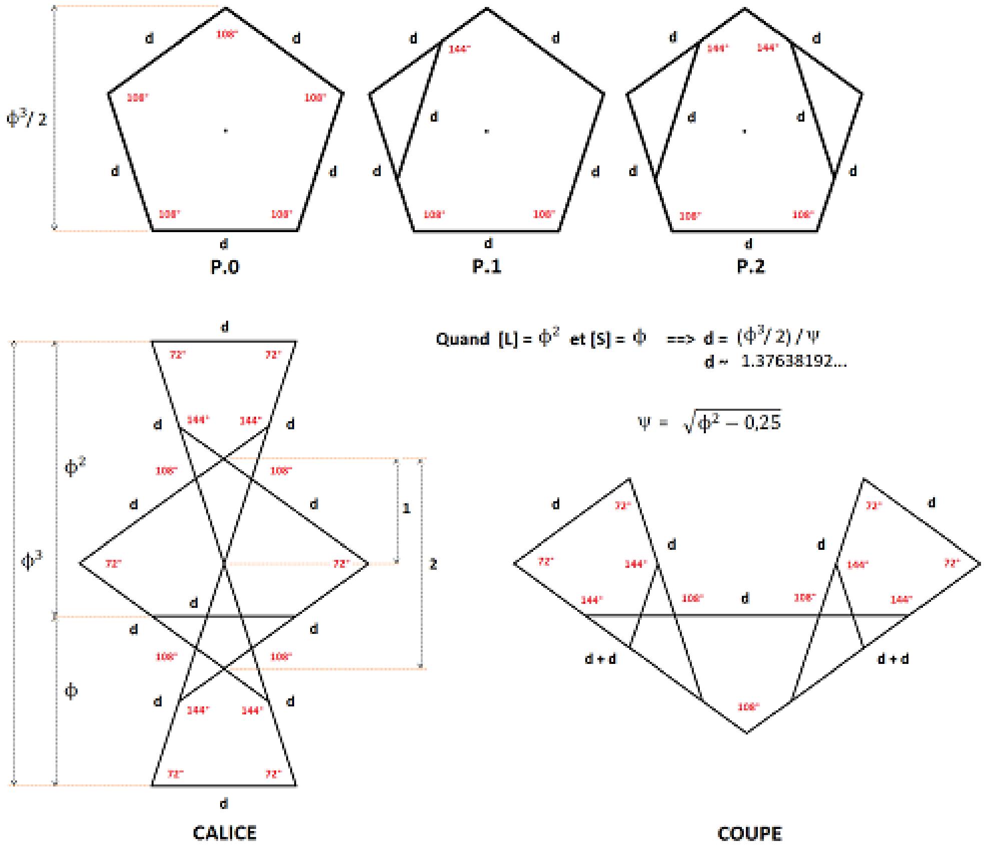

Figure 4.

When [L] = Φ^2 and [S] = Φ ==> $d = (\Phi^3/2)/\Psi$; $d \sim 1.37638192...$; $\Psi = \sqrt{(\Phi^2\text{-}0,25)}$

Conclusion :

This article shows that there are substitution rules (or morphisms), in keeping with the infinite Fibonacci word, producing only palindromes from a binary alphabet : $\boldsymbol{A}$ = {L, S} ; the concatenation of the infinity of these palindromes is the infinite Fibonacci word. Each palindrome generated by this morphism can be superposed exactly on the diameter of the circles forming a geometric sequence with common ratio q = Φ and therefore a fractal system of circles. Thus, we can locate the origin of this sequence in a point (O.) situated at -∞ ; this point being, by definition, physically unattainable. The fractal system of circles creates 36° and 72° angles, which implies ten identical systems over the 360° of the plane space. The point of origin therefore becomes the center of a perfect pentagonal and decagonal symmetry on the infinity of the plane and thus a two-dimensional quasicrystal. The “pentagrid” made up of the parallel lines, the spacings distribution of which obeys the development of the Fibonacci words and is centered on the origin, evens out very precisely despite the irrational nature of the four different segment lengths it is made up of and their repetitions over the infinity of the plane.

This article approaches the Fibonacci sequence and infinite word in a brand-new way ; it sheds light on a graphic logic and a geometric development which answer some questions of material physics, such as growth and organization of atoms in quasicrystals.

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